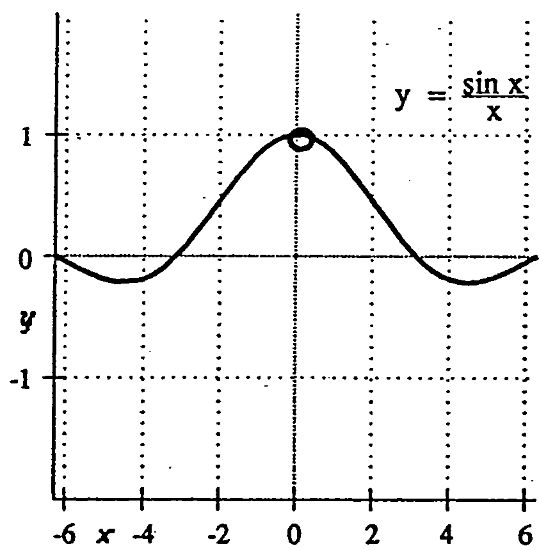
f is continuous at $x = a$ if and only if

$$\lim_{x \to a} f(x) = f(a)$$

LIMITS,

CONTINUITY,

and other humorous stories



$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

TABLE OF CONTENTS

INFORMAL DEFINITION OF A LIMIT

$$\lim_{x \to a} f(x) = L$$

implies that as x gets "closer and closer" to the number a from either side, *without actually equaling* a, f(x) gets "closer and closer" to the number L.

We will leave the concept "closer and closer" as an intuitive one. There is a more formal and precise definition of limits, which you will learn if you continue your studies of mathematics in college. I state it here in its full mathematical splendor merely as a preview of things to come. We will not be using it in this course.

FORMAL DEFINITION OF A LIMIT

$$\lim_{x \to a} f(x) = L \text{ iff } \forall\ \varepsilon > 0, \exists\ \delta > 0, \ni\ 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

It is extremely important to remember in our informal, intuitive definition that we only let x approach, but never equal a. We make no statement about the value of the function at x = a. f(a) may not even exist, but the limit may exist! It will seem that in our techniques for finding limits, we will be "dividing by 0". Not so! x will only be forever approaching, but never equaling a.

REPEAT 100 TIMES

The limit as x -> a of f(x) may exist even though f(a) may be undefined.

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And one more time for good measure!

The limit as x -> a of f(x) may exist even though f(a) may be undefined.

Theorems on Limits

If c, k, R, S, U, and V are finite numbers and if

$$\lim_{x\to c} f(x) = R, \quad \lim_{x\to c} g(x) = S, \quad \lim_{x\to\infty} f(x) = U, \quad \lim_{x\to\infty} g(x) = V,$$

then:

(1a)	$\lim_{x\to c} kf(x) = kR$	**(1b)**	$\lim_{x\to\infty} kf(x) = kU$
(2a)	$\lim_{x\to c} [f(x) + g(x)] = R + S$	**(2b)**	$\lim_{x\to\infty} [f(x) + g(x)] = U + V$
(3a)	$\lim_{x\to c} f(x)g(x) = RS$	**(3b)**	$\lim_{x\to\infty} f(x)g(x) = UV$
(4a)	$\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{R}{S}$ (if $S \neq 0$)	**(4b)**	$\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{U}{V}$ (if $V \neq 0$)
(5a)	$\lim_{x\to c} k = k$	**(5b)**	$\lim_{x\to\infty} k = k$

(6) (The "Squeeze" Theorem.) If $f(x) \leqq g(x) \leqq h(x)$ and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

TECHIQUES FOR FINDING LIMITS

In general, to find the limit of f(x) as x approaches a, your first instinct should be to find f(a). If f(a) = L, where L is a real number, that is the limit.

Examples:

$$\lim_{x \to -1} \frac{x - 1}{x^2 + 1} = \frac{-1 - 1}{(-1)^2 + 1} = \frac{-2}{2} = -1$$

$$\lim_{x \to 0} \frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$$

However, if $f(a) = \frac{k}{0}$, k a nonzero number, then the limit does not exist.

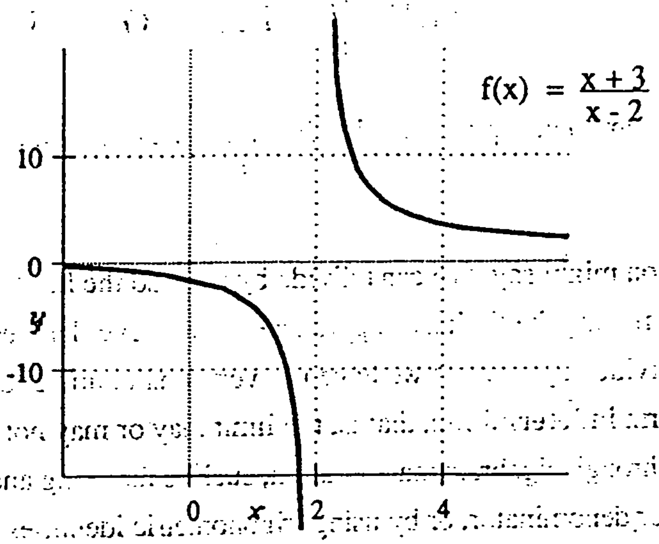
Examples:

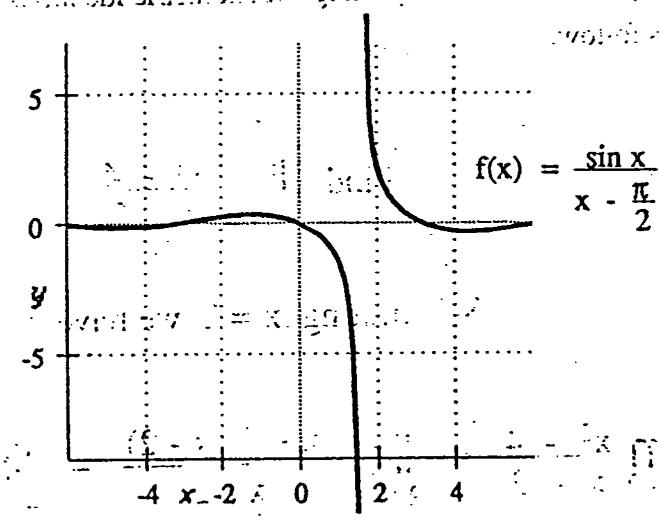
$$\lim_{x \to 2} \frac{x + 3}{x - 2} = \frac{5}{0} = \infty$$

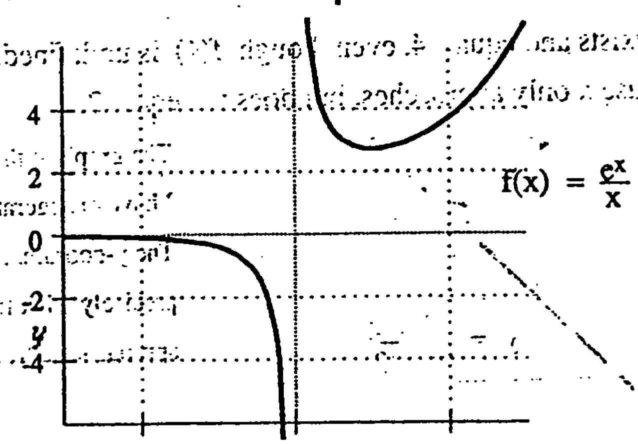
$$\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{x - \frac{\pi}{2}} = \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2} - \frac{\pi}{2}} = \frac{1}{0} = \infty$$

$$\lim_{x \to 0} \frac{e^x}{x} = \frac{e^0}{0} = \frac{1}{0} = \infty$$

The graphs of the functions used in the previous 3 examples are shown below. Notice that in each case there is a vertical asymptote at the point where we are attempting to find the limit.







INDETERMINATE FORM

THE $\frac{0}{0}$ TYPE OF LIMIT

So far, limits seem to be a piece of cake. Just plug and chug. But what if we plug in and get

$$\frac{0}{0}$$

No problem, you might say. We can't divide by zero, so the limit is undefined. Ah, but remember we said that in finding the limit as x approaches a, we never let x equal exactly a. So we never really have 0 divided by 0. What we have is a very small number divided by a very small number. We call this form **indeterminite**, that is, the limit may or may not exist. We must investigate further, either through algebraic manipulation, such as factoring and canceling, or by rationalizing the numerator or denominator, or by using trigonometric identities and some well-known limits. Some examples follow.

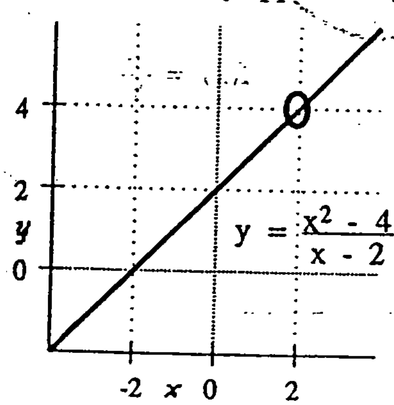
Example 1.

Find $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

Substituting $x = 2$ we have $\frac{0}{0}$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} x + 2 = 4$$

Thus, the limit exists and equals 4, even though $f(4)$ is undefined. Note that it was okay to divide by $(x - 2)$ because x only approaches, but does not equal 2.



The graph of the function is shown at left.
I have exaggerated the point of discontinuity at x = 2. The y-coordinate of the "hole" is 4, and that is precisely what the limit of the function is as x approaches 2 from either side.

Example 2.

$$\text{Find} \lim_{x \to 0} \frac{x^2 - x}{x^3 - x^2}$$

Again, substituting $x = 0$, we get $\frac{0}{0}$

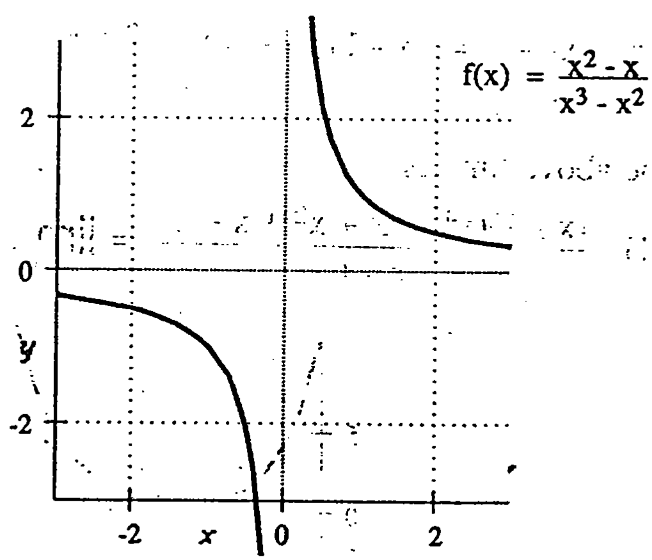
$$\lim_{x \to 0} \frac{x^2 - x}{x^3 - x^2} = \lim_{x \to 0} \frac{x(x - 1)}{x^2(x - 1)} = \lim_{x \to 0} \frac{1}{x} = \frac{1}{0} = \infty$$

Here, the limit does not exist.

The graph of the function is shown below. We see that it is exactly like

$$f(x) = \frac{1}{x}$$

and has a vertical asymptote at $x = 0$.



Example 3.

$$\text{Find } \lim_{x \to 1} \frac{x^5 - 1}{x - 1}$$

$$f(1) = \frac{0}{0}$$

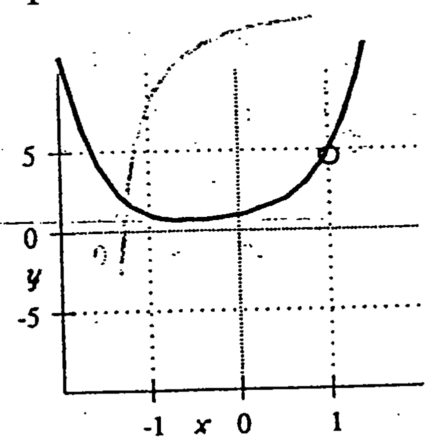
So whatever factor is making the denominator 0, is also making the numerator 0, i.e. (x - 1). We use synthetic division to factor the numerator.

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ & & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & \boxed{0} \end{array}$$

Therefore, $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$

Thus, the above limit is

$$\lim_{x \to 1} \frac{(x - 1)(x^4 + x^3 + x^2 + x + 1)}{x - 1} = \lim_{x \to 1} x^4 + x^3 + x^2 + x + 1 = 5$$



$$y = \frac{x^5 - 1}{x - 1}$$

The graph of the function is shown above. There is a point discontinuity at $x = 1$. The y-coordinate of the "hole" is the same as the limit, 5.

Example 4.

$$\text{Find } \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$\lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

Dividing every term by h we have

$$\lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$$

Note that the above limit is the definition of the derivative of $f(x) = x^3$. Recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Thus, if $f(x) = x^3$, then

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = 3x^2$$

Example 5.

$$\text{Find} \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

We factor out $\frac{1}{h}$ and combine the fractions in the numerator.

$$\lim_{h \to 0} \frac{1}{h} \left\{ \frac{1}{x+h}\frac{x}{x} - \frac{1}{x}\frac{x+h}{x+h} \right\} = \lim_{h \to 0} \frac{1}{h} \left\{ \frac{x - (x+h)}{x(x+h)} \right\} =$$

$$\lim_{h \to 0} \frac{1}{h} \left\{ \frac{-h}{x(x+h)} \right\} = \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

By the definition of the derivative on the previous page, we have just found the derivative of $f(x) = \frac{1}{x}$.

$$\text{If } f(x) = \frac{1}{x}, \text{ then } f'(x) = -\frac{1}{x^2}$$

Note that $f'(x) = -\frac{1}{x^2}$ is always negative for $x \neq 0$

Recall that a derivative is a slope. From the graph of $f(x) = \frac{1}{x}$ below, we see that it is falling everywhere to the right where it is defined. Therefore its slope, and its derivative, is negative for all $x \neq 0$.

Also note that the where f(x) is undefined, so is its derivative. This is true for all functions.

Example 6.

Find $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

The limit is of the $\frac{0}{0}$ type. We rationalize the numerator.

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \lim_{x \to 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} =$$

$$\lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2 + 2} = \frac{1}{4}$$

Example 7.

Find $\lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}$

Note that the above is the definition of the derivative of $f(x) = \sqrt{x}$

Once again, we rationalize the numerator.

$$\lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} =$$

$$\lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x + h} + \sqrt{x})} =$$

$$\lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

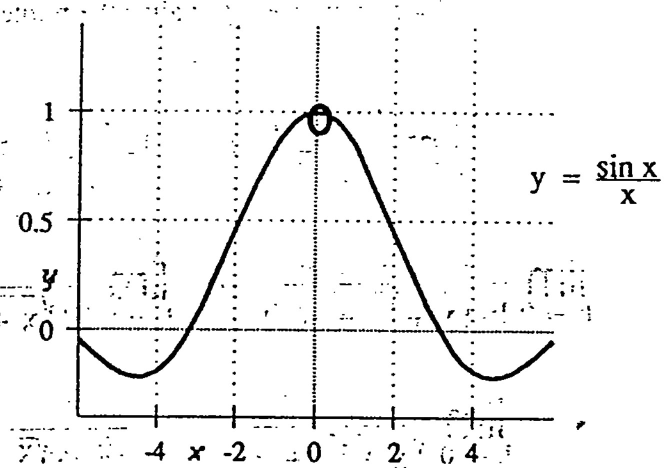
$\therefore$ if $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$

LIMITS OF TRIGONOMETRIC FUNCTIONS

A FIVE STAR LIMIT (* * * * * Count them. Five!)

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

We formally derived this limit last yaer using the unit circle, the area of a circular sector, trigonometric ratios, and the Squeeze Theorem. Since each generation should not have to re-invent the wheel, we shall be content this time to merely sketch the graph of the function and note that as we approach 0 from either side, f(x) approaches 1, although f(0) is undefined. Once again, I exaggerate the point discontinuity at x = 0.



Also recall that the above limit is 1 only if θ is measured in radians. If we use degrees, then the limit becomes .01745329252... Since this limit is involved in many other limits, and in the derivatives of all the trigonometric functions, we would have to carry this cumbersome number in our pocket at all times. Thus, the derivative of sin x would be .01745329252(cos x), rather than just cos x. We legislate our way out of this mess by decreeing we shall only use radians. Why not? After all, it's our coloring book, and we can color the bunnies any way we want!

Techniques for Finding Limits of Trigonometric Functions

We use the 5-Star limit on the previous page, in conjunction with trigonometric identities and algebraic manipulation, to find many other trigonometric limits.

Example 1.

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{1}{x}\frac{\sin x}{\cos x} = \lim_{x \to 0} \frac{\sin x}{x}\frac{1}{\cos x} = \left(\lim_{x \to 0} \frac{\sin x}{x}\right)\left(\frac{1}{\cos 0}\right) = (1)(1) = 1$$

It might be worth remembering that the above limit is 1, since it comes up frequently.
Note that the above implies that for small x, x in radians,

$\tan x \approx x$. The same holds true for $\sin x$. For small x (in radians), $\sin x \approx x$

Example 2.

$$\lim_{x \to 0} x \csc x = \lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}} = \frac{1}{1} = 1$$

Of course, we could have merely noted that since $\frac{x}{\sin x}$ is the reciprocal of $\frac{\sin x}{x}$

its limit as x approaches 0 is the reciprocal of $\lim_{x \to 0} \frac{\sin x}{x}$

and the reciprocal of 1 is 1.

Example 3.

$$\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{2 \sin x \cos x}{x} = \lim_{x \to 0} 2 \cos x \frac{\sin x}{x} = 2(\cos 0)(1) = 2(1)(1) = 2$$

There is an alternate way of finding the above limit.

$$\lim_{x \to 0} \frac{2}{2} \frac{\sin 2x}{x} = \lim_{x \to 0} 2 \frac{\sin 2x}{2x}$$

Now let $\theta = 2x$. Then as $x \to 0$, $\theta \to 0$. Substituting $\theta = 2x$ in the expression above, we have

$$\lim_{x \to 0} 2 \frac{\sin 2x}{2x} = \lim_{\theta \to 0} 2 \frac{\sin \theta}{\theta} = 2(1) = 2$$

Example 4.

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin (x - \frac{\pi}{2})}{x - \frac{\pi}{2}}$$

Note that $f(\frac{\pi}{2}) = \frac{0}{0}$. Make the substitution $\theta = x - \frac{\pi}{2}$.

Then as $x \to \frac{\pi}{2}$, $\theta \to 0$.

$$\therefore \lim_{x \to \frac{\pi}{2}} \frac{\sin (x - \frac{\pi}{2})}{x - \frac{\pi}{2}} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Pretty neat, huh?

Example 5.

$$\text{Find } \lim_{x \to 0} \frac{\sin 2x}{\sin 3x}$$

The tempation may be very strong to cancel the sins and cancel the x's. There's no mathematical rhyme or reason for that, of course, but due to a fantastic set of circumstances, coincidences, conjunction of the planets and the stars and other astrological influences, not to mention sheer luck, you would be right!

The answer is indeed $\frac{2}{3}$!

Let us see why. We begin by performing the usual mathematical miracle of multiplying by 1, not once, but twice.

$$\lim_{x \to 0} 2x \frac{\sin 2x}{2x} \frac{1}{3x} \frac{3x}{\sin 3x} = \lim_{x \to 0} \frac{2x}{3x} \frac{\sin 2x}{2x} \frac{3x}{\sin 3x} =$$

$$\lim_{x \to 0} \frac{2}{3} \frac{\sin 2x}{2x} \frac{3x}{\sin 3x}$$

Now if we let $\theta = 2x$ and $\varphi = 3x$, then as $x \to 0$, both $\theta, \varphi \to 0$, and we have

$$\lim_{\theta,\varphi \to 0} \frac{2}{3} \frac{\sin \theta}{\theta} \frac{\varphi}{\sin \varphi} = \frac{2}{3}(1)(1) = \frac{2}{3}$$

Now isn't that more satisfying then blindly canceling the sins and the x's ? No, huh ? Very well, do it your way. In general

$$\lim_{x \to 0} \frac{\sin mx}{\sin nx} = \frac{m}{n}, \; n \neq 0$$

Example 6. We will need the following limit when we derive the derivative of sin x. Again note that it's a 0 over 0 type.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x} \frac{1 + \cos x}{1 + \cos x} = \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} =$$

$$\lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \to 0} \frac{\sin x}{x} \frac{\sin x}{1 + \cos x} = (1)\left(\frac{\sin 0}{1 + \cos 0}\right) = (1)\left(\frac{0}{1+1}\right) = 0$$

Some LEMMING types of limits.

$$\lim_{x \to 1} 5 = 5$$

5 is a constant. As x approaches 1, 5 does not change. The limit is 5, not 1.

In general, if k is a constant, $\lim_{x \to a} k = k$

$$\lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1$$

$$\lim_{x \to 0} \frac{\cos x}{x} = \frac{\cos 0}{0} = \frac{1}{0} = \infty, \quad \textbf{not } 1\,!$$

ONE-SIDED LIMITS

$$\lim_{x \to a^-} f(x) = L_1$$

means that f(x) gets closer and closer to L_1 as x approaches a from the left.

$$\lim_{x \to a^+} f(x) = L_2$$

means that f(x) gets closer and closer to L_2 as x approaches a from the right.

EXAMPLE:

$$\text{Let} \quad f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 3 - x, & x > 1 \end{cases}$$

$$\text{Then} \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + 2 = 3$$

$$\text{and} \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3 - x = 2$$

Note that for the above function at $x = 1$, the left limit does not equal the right limit. We say that the limit at $x = 1$ does not exist.

$\lim_{x \to a} f(x)$ **exists if and only if**

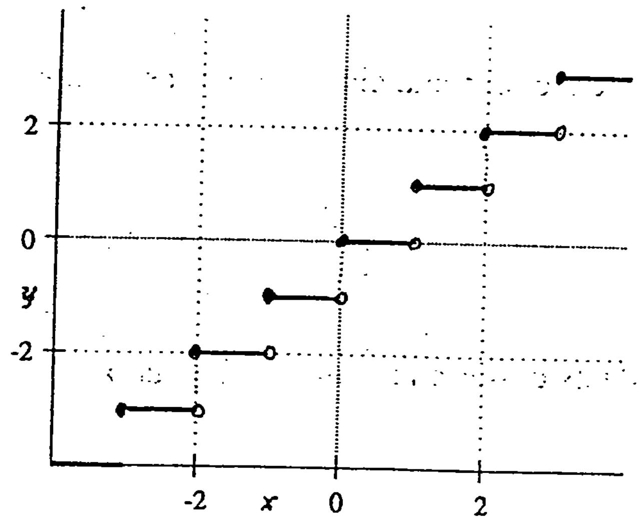
$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

THE GREATEST INTEGER FUNCTION $f(x) = [x]$

$[x]$ is defined as the greatest integer less than or equal to x

$[3.4] = 3$ $[3.99] = 3$ $[2] = 2$ $[0.8] = 0$ $[-1.4] = -2$

The function is graphed below.



$$\lim_{x \to 2^-} [x] = 1$$

$$\lim_{x \to 2^+} [x] = 2$$

Therefore, $\lim_{x \to 2} [x]$ **is non-existent because**

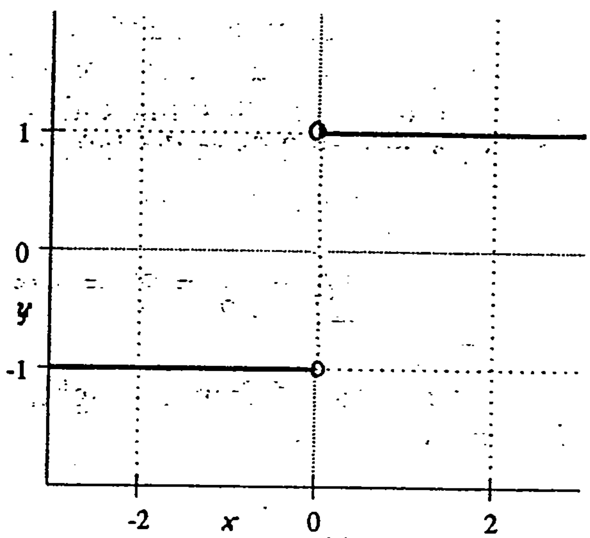
$$\lim_{x \to 2^-} [x] \neq \lim_{x \to 2^+} [x]$$

In general, if a is an integer, then

$\lim_{x \to a} [x]$ **does not exist.**

THE STEP FUNCTION

The graph of $f(x) = \frac{|x|}{x}$ is shown below.



Note that for x > 0, f(x) = 1; for x < 0, f(x) = -1; and f is undefined at x = 0.

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = +1$$

$$\therefore \lim_{x \to 0} \frac{|x|}{x} \text{ is non-existent.}$$

However, $\lim_{x \to 2} \frac{|x|}{x} = 1$ and $\lim_{x \to -3} \frac{|x|}{x} = -1$