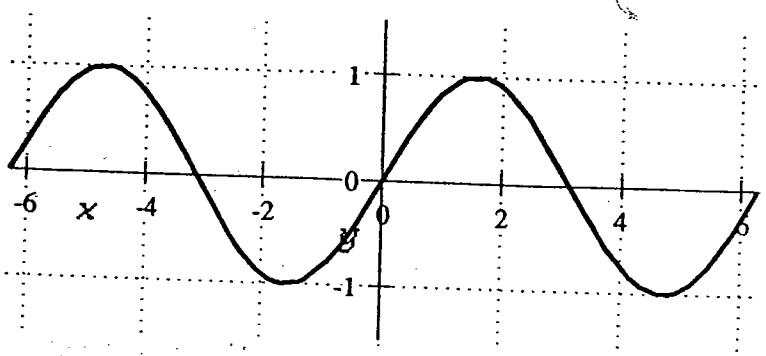
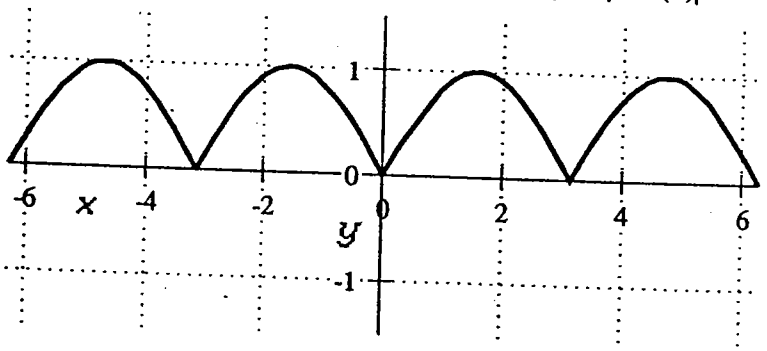
PROPERTIES OF FUNCTIONS

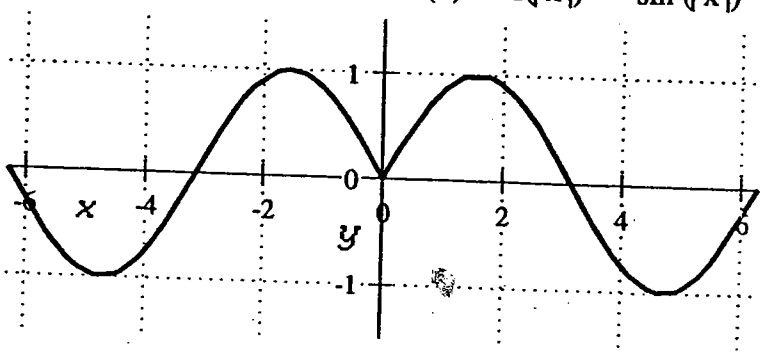
$f(x) = \sin(x)$



$g(x) = |f(x)| = |\sin(x)|$



$h(x) = f(|x|) = \sin(|x|)$



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GRAPHS OF ELEMENTARY FUNCTIONS

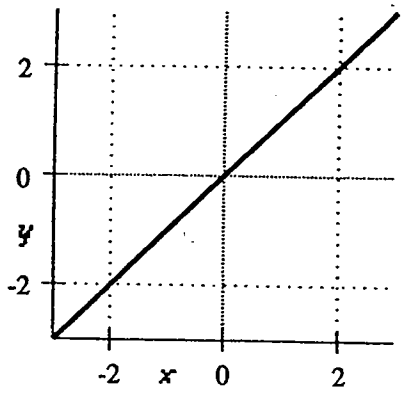
$f(x) = x$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Intercepts: (0, 0)

Symmetry: origin



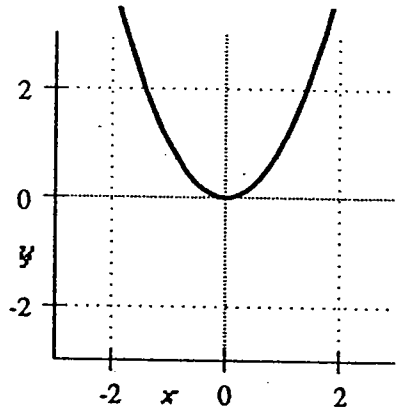
$f(x) = x^2$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Intercepts: (0,0)

Symmetry: y-axis



$f(x) = x^n$,

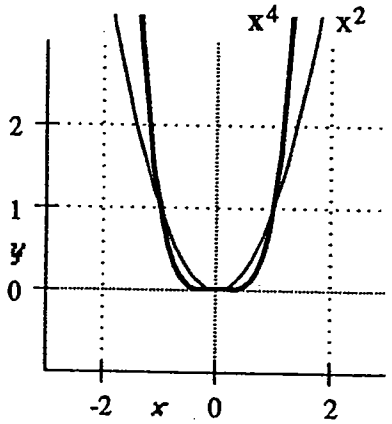
n an even positive integer

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Intercepts: (0, 0)

Symmetry: y-axis



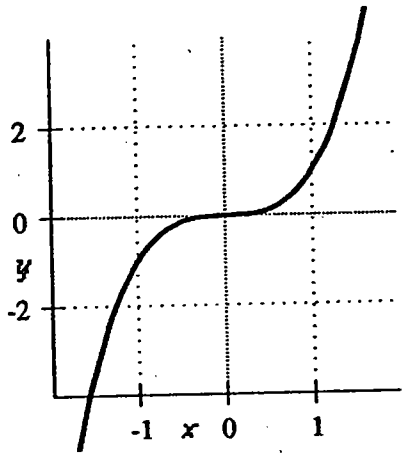
$f(x) = x^3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Intercepts: (0, 0)

Symmetry: origin



$f(x) = x^n$

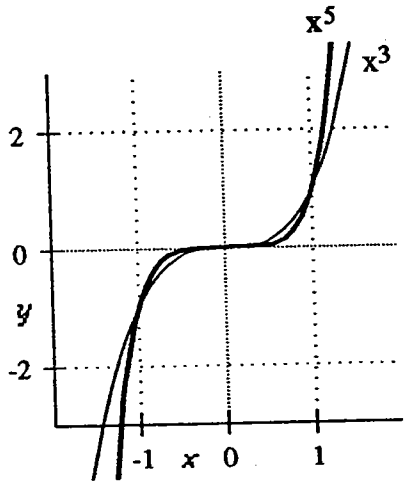
n a positive odd integer

Domain: $(-\infty,\infty)$

Range: $(-\infty,\infty)$

Intercepts: (0,0)

Symmetry: origin

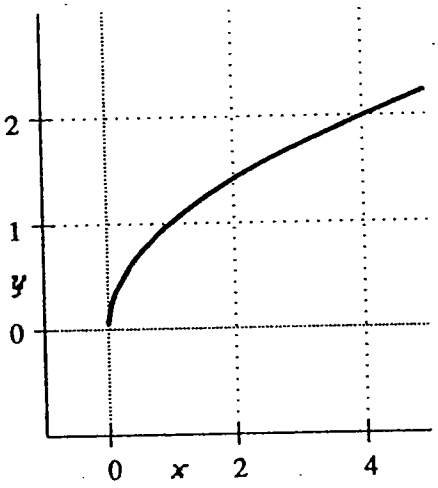


$f(x) = \sqrt{x}$

Domain: $[0, \infty)$

Range: $[0, \infty)$

Intercepts: (0, 0)



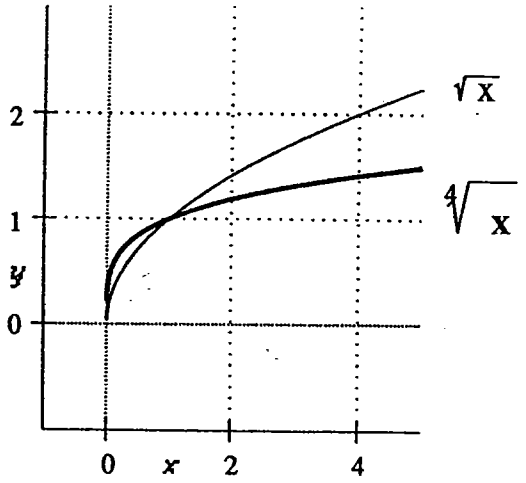
$f(x) = \sqrt[n]{x}$

n an even positive integer

Domain: $[0, \infty)$

Range: $[0, \infty)$

Intercepts: (0, 0)



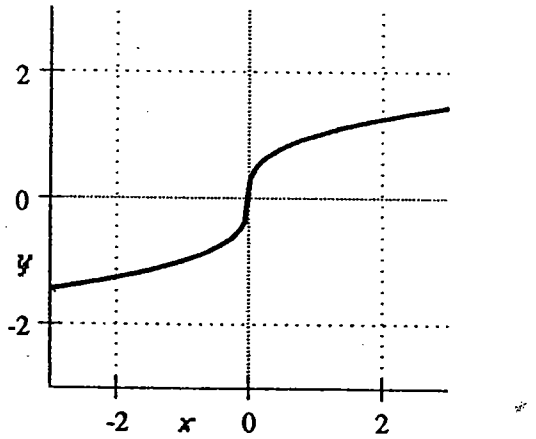
$f(x) = \sqrt[3]{x}$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Intercepts: (0, 0)

Symmetry: origin



$f(x) = \sqrt[n]{x}$

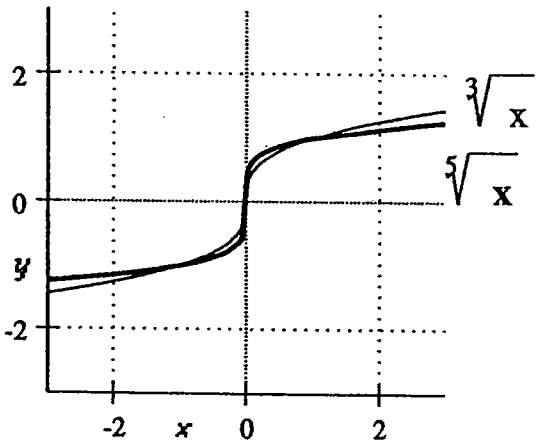
n a positive odd integer

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

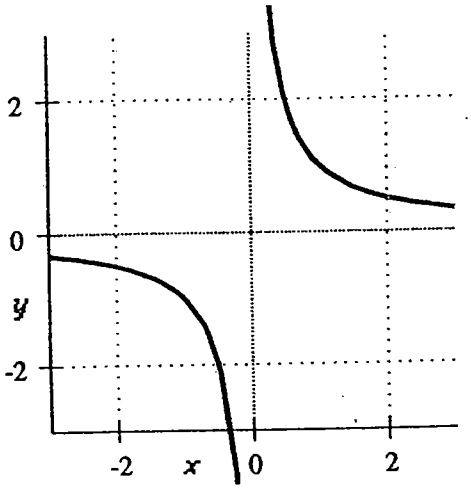
Intercepts: (0, 0)

Symmetry: origin



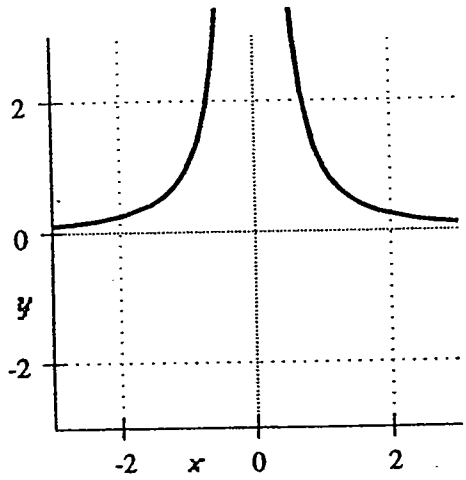
$$f(x) = \frac{1}{x}$$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
Intercepts: none
Symmetry: origin
Asymptotes: $x = 0, y = 0$



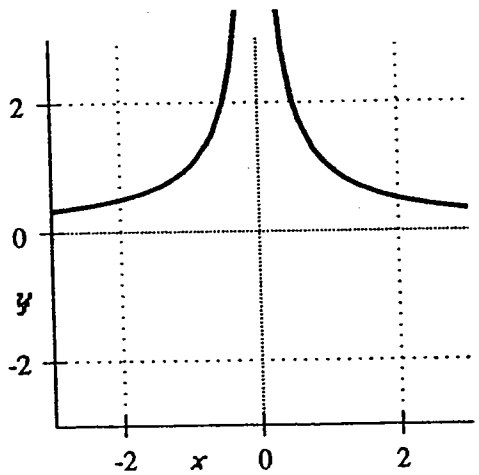
$$f(x) = \frac{1}{x^2}$$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$
Intercepts: none
Symmetry: y-axis
Asymptotes: $x = 0, y = 0$



$$f(x) = \frac{1}{|x|}$$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$
Intercepts: none
Symmetry: y-axis
Asymptotes: $x = 0, y = 0$



$f(x) = \sqrt{a^2 - x^2}$

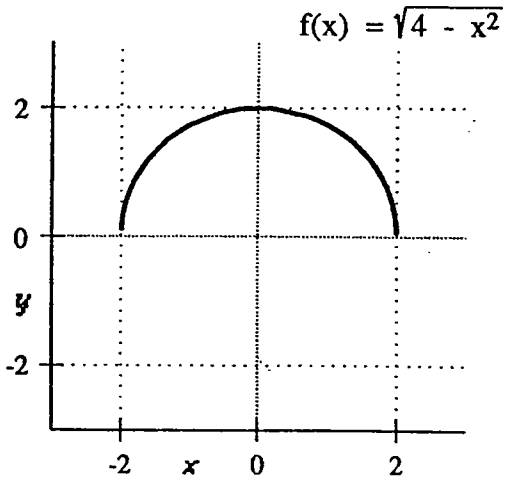
Semi-circle of radius a

Domain: [-a, a]

Range: [0, a]

Intercepts: (-a, 0), (a, 0), (0, a)

Symmetry: y-axis



$f(x) = \sqrt{x^2 - a^2}$

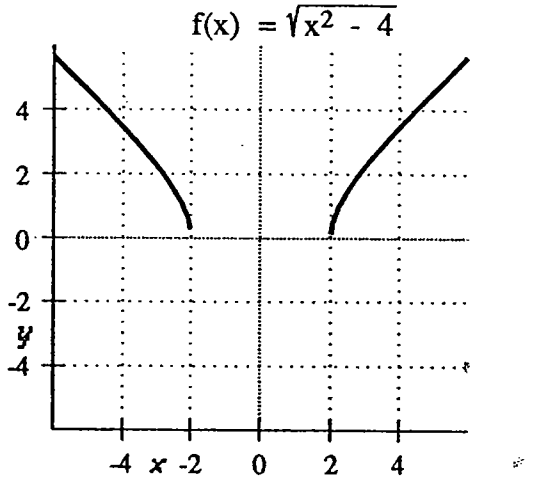
positive part of a hyperbola

Domain: $(-\infty, -a] \cup [a, \infty)$

Range: $[0, \infty)$

Intercepts: (-a, 0), (a, 0)

Symmetry: y-axis



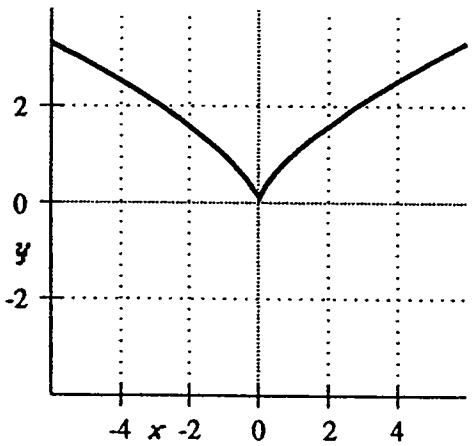
$f(x) = x^{\frac{2}{3}}$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Intercepts: (0, 0)

Symmetry: y-axis



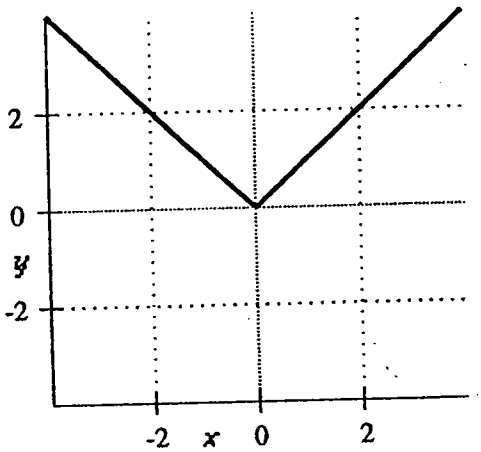
$f(x) = |x|$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Intercepts: (0, 0)

Symmetry: y-axis



$f(x) = \frac{|x|}{x}$

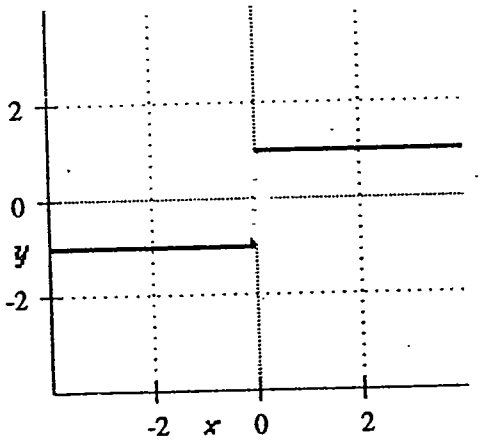
Step function

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $y = -1, y = 1$ only

Intercepts: None

Symmetry: origin



$f(x) = [x]$

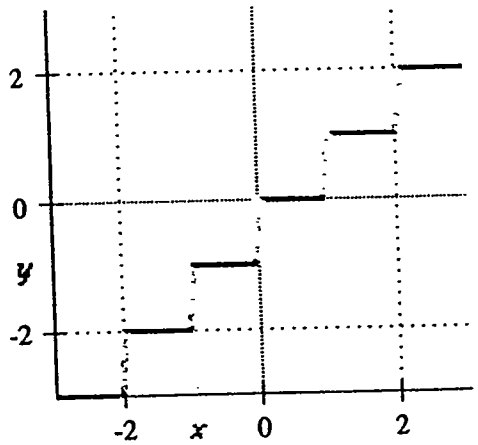
Greatest integer $\leq x$

Domain: $(-\infty, \infty)$

Range: the set of all integers

Intercepts: (0, 0)

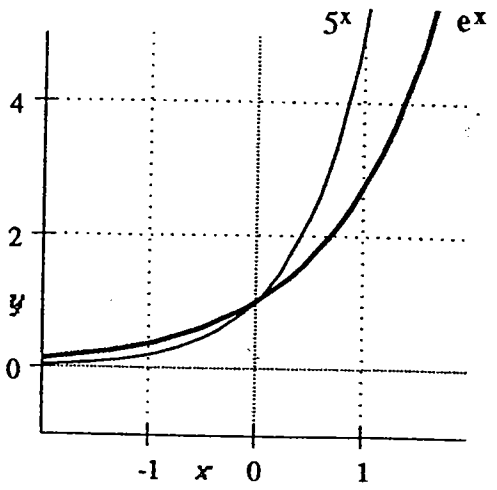
Symmetry: none



$f(x) = b^x \quad b > 1$

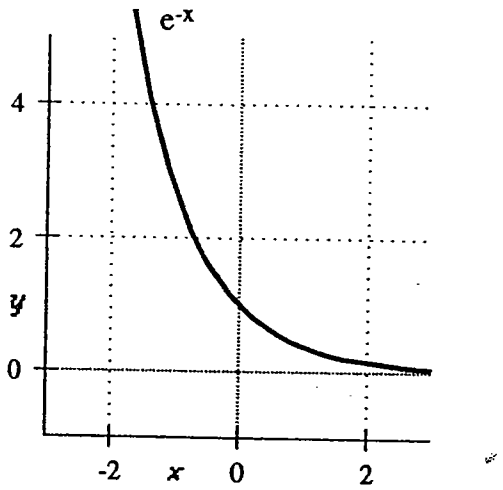
exponential function

Domain:	$(-\infty, \infty)$
Range:	$(0, \infty)$
Intercepts:	$(0, 1)$
Symmetry:	none
Asymptotes:	$y = 0$



$f(x) = b^{-x}, \quad b > 1$

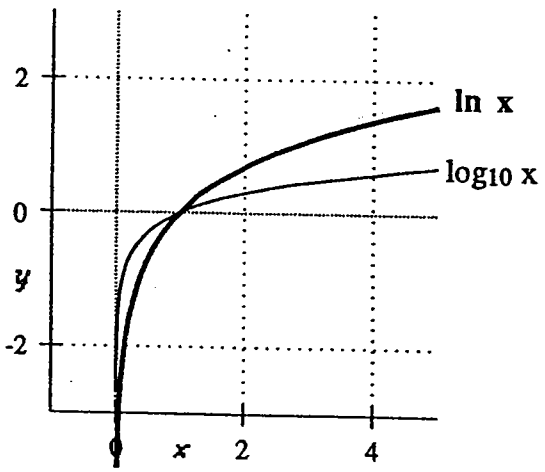
Domain:	$(-\infty, \infty)$
Range:	$(0, \infty)$
Intercepts:	$(0, 1)$
Symmetry:	none
Asymptotes:	$y = 0$



$f(x) = \log_b x, \quad b > 1$

logarithmic function

Domain:	$(0, \infty)$
Range:	$(-\infty, \infty)$
Intercepts:	$(1, 0)$
Symmetry:	none
Asymptotes:	$x = 0$



$f(x) = \sin x$

Domain: $(-\infty, \infty)$

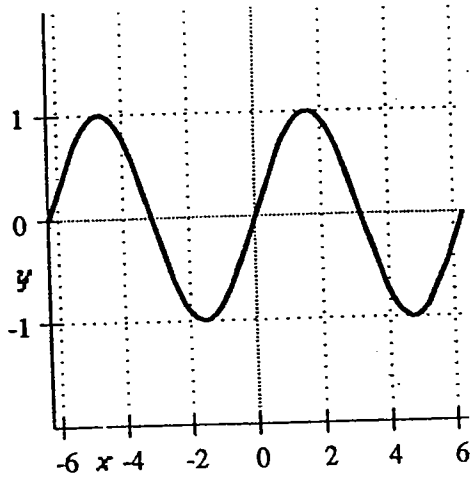
Range: $[-1, 1]$

Intercepts: $x = n\pi$, n an integer

$y = 0$

Symmetry: origin

Period: 2π



$f(x) = \cos(x)$

Domain: $(-\infty, \infty)$

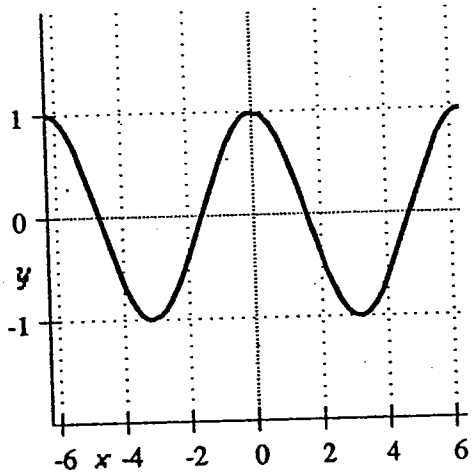
Range: $[-1, 1]$

Intercepts: $x = (2n+1)\frac{\pi}{2}$

$y = 1$

Symmetry: y-axis

Period: 2π



$f(x) = \tan x$

Domain: $x \neq (2n+1)\frac{\pi}{2}$

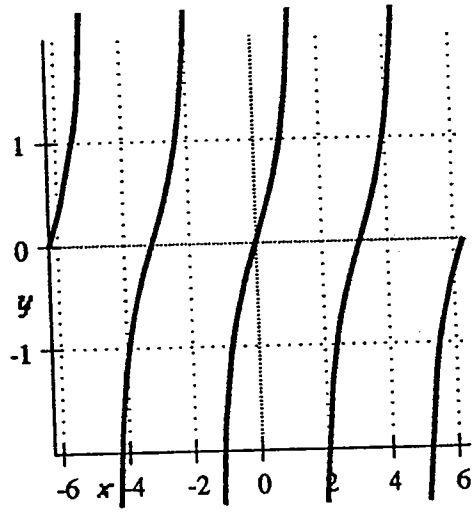
Range: $(-\infty, \infty)$

Intercepts: $x = n\pi$

$y = 0$

Symmetry: origin

Period: π



$$f(x) = \csc x = \frac{1}{\sin x}$$

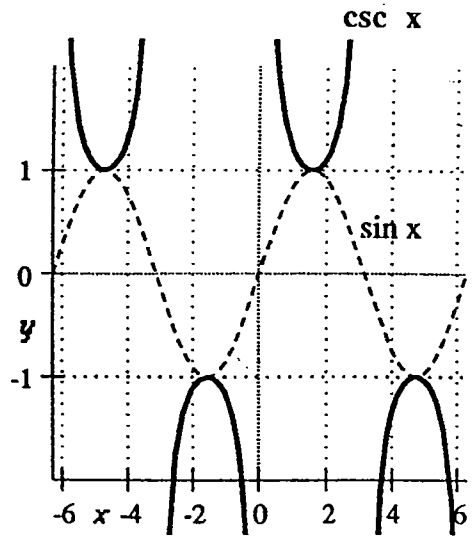
Domain: $x \neq n\pi$

Range: $|y| \geq 1$

Intercepts: none

Symmetry: origin

Period: 2π



$$f(x) = \sec x = \frac{1}{\cos x}$$

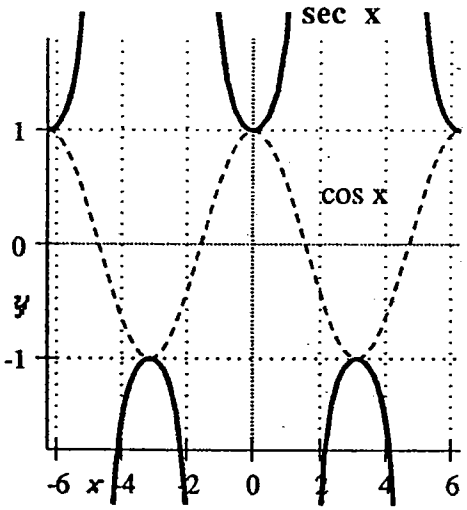
Domain: $x \neq (2n+1)\frac{\pi}{2}$

Range: $|y| \geq 1$

Intercepts: $y = 1$

Symmetry: y-axis

Period: 2π



$$f(x) = \cot x = \frac{1}{\tan x}$$

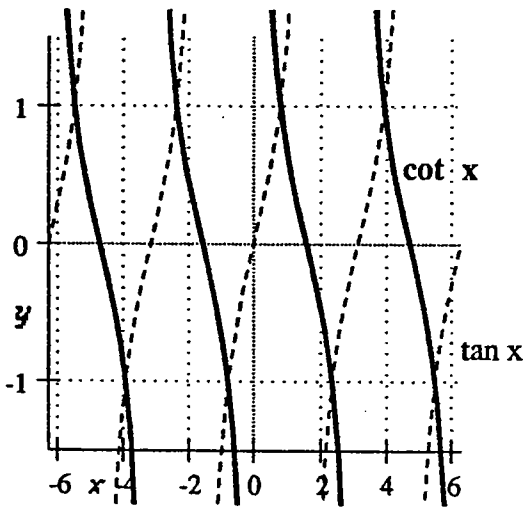
Domain: $x \neq n\pi$

Range: $(-\infty, \infty)$

Intercepts: $x = (2n+1)\frac{\pi}{2}$

Symmetry: origin

Period: π



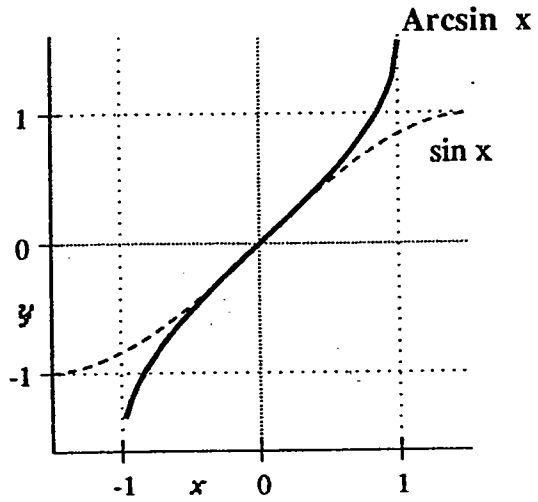
f(x) = Arcsin x

Domain: [-1, 1]

Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Intercepts: (0, 0)

Symmetry: origin



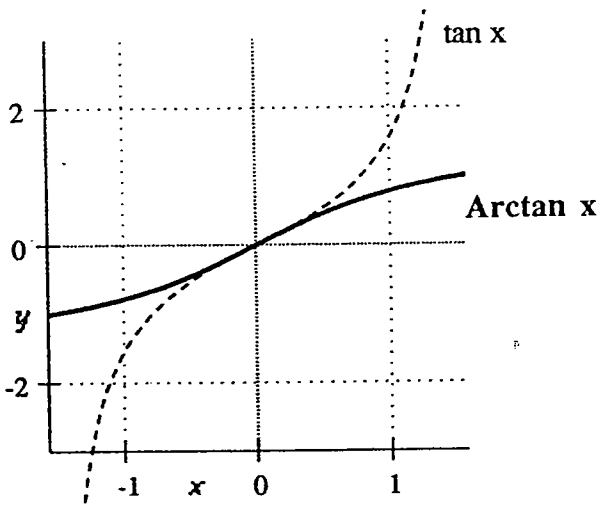
f(x) = Arctan x

Domain: $(-\infty, \infty)$

Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$

Intercepts: (0, 0)

Symmetry: origin



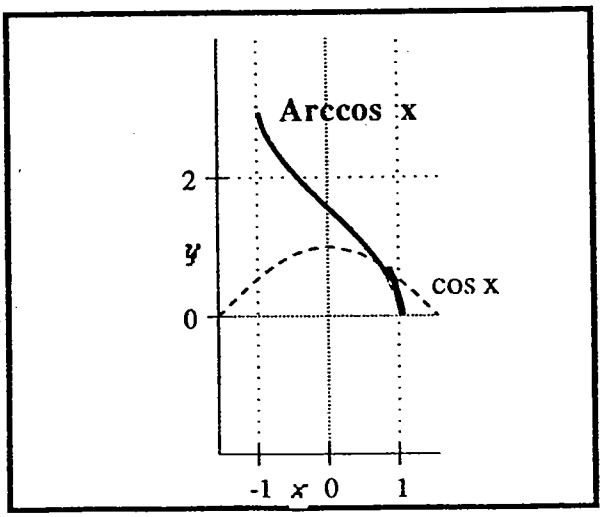
f(x) = Arccos x

Domain: [-1, 1]

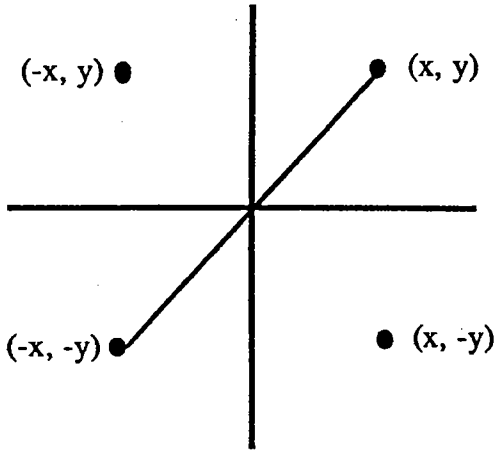
Range: $[0, \pi]$

Intercepts: $(0, \frac{\pi}{2})$

Symmetry: None



SYMMETRY



The diagram above shows the necessary conditions for symmetry about the axes and the origin.

The graph of a relation is symmetric to

1. **the y-axis if for all points (x, y) on its graph, the point (-x, y) is also on the graph.**
2. **the x-axis if for all points (x, y) on its graph, the point (x, -y) is also on the graph.**
3. **the origin if for all points (x, y) on its graph, the point (-x, -y) is also on the graph.**

For symmetry to the axes, the axis can be regarded as the surface of a mirror. For point symmetry to the origin, if we draw a line from the point (x, y) through the origin, the point (-x, -y) will lie on the line an equal distance on the other side of the origin.

RULES FOR DETERMINING SYMMETRY

The graph of a relation is symmetric to

1. **the y-axis if replacing x by -x yields an equivalent relation.**
2. **the x-axis if replacing y by -y yields an equivalent relation.**
3. **the origin if replacing x by -x and y by -y yields an equivalent relation.**

EXAMPLE. Determine the symmetry of the following relations:

a. $y = 3x^2 - 2$

b. $y = 4x^3 - 1$

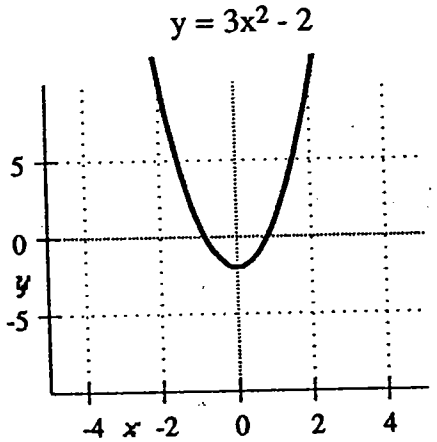
c. $y = x^3 - x$

d. $x^2 + y^2 = 1$

a. To y-axis $y = 3(-x)^2 - 2 \quad \rightarrow \quad y = 3x^2 - 2$
YES

To x-axis $-y = 3x^2 - 2 \quad \rightarrow \quad y = -(3x^2 - 2)$
NO

To origin $-y = 3(-x)^2 - 2 \quad \rightarrow \quad -y = 3x^2 - 2$
NO



Symmetry to the y-axis

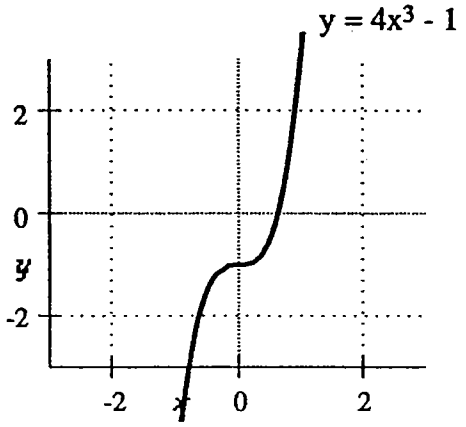
b. $y = 4x^3 - 1$

To y-axis: $y = 4(-x)^3 - 1$ --> $y = -4x^3 - 1$
No.

To x-axis: $-y = 4x^3 - 1$ --> $y = -4x^3 + 1$
No.

To origin: $-y = 4(-x)^3 - 1$ --> $y = 4x^3 + 1$
No.

No symmetry.



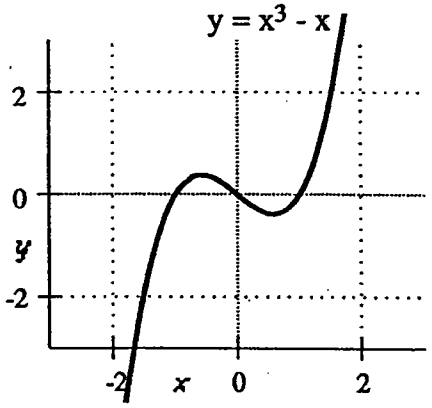
c. $y = x^3 - x$

To y-axis: $y = (-x)^3 - (-x)$ --> $y = -x^3 + x$
No.

To x-axis $-y = x^3 - x$ --> $y = -x^3 + x$
No.

To origin: $-y = (-x)^3 - (-x)$ --> $y = x^3 - x$
Yes.

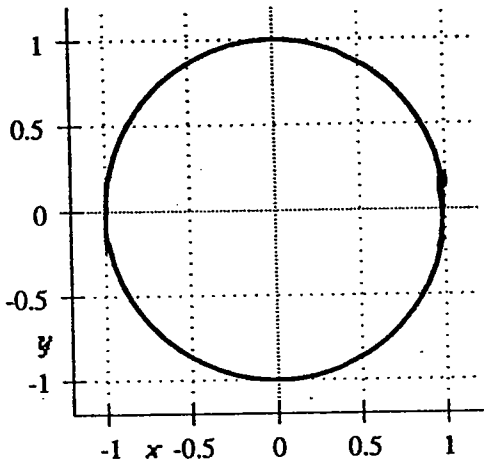
Symmetry to origin



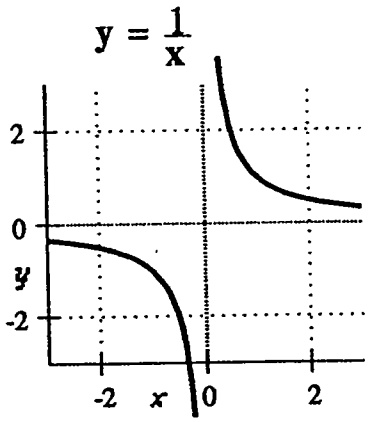
d. $x^2 + y^2 = 1$

To y-axis: $(-x)^2 + y^2 = 1$ --> $x^2 + y^2 = 1$
Yes.
To x-axis: $x^2 + (-y)^2 = 1$ --> $x^2 + y^2 = 1$
Yes.
To origin: $(-x)^2 + (-y)^2 = 1$ --> $x^2 + y^2 = 1$
Yes.

Symmetric to both axes and origin.



Note that if the graph of a relation is symmetric to both the x and y-axes, it is symmetric to the origin. The converse is not true. A relation may be symmetric to the origin, but not to the axes. Consider the graph of



The graph is symmetric to the origin, but not to either axes.

ODD AND EVEN FUNCTIONS

DEFINITION

1. **A function is said to be even if, for all x in its domain, $f(-x) = f(x)$.**

2. **A functions is said to be odd if, for all x in its domain, $f(-x) = -f(x)$.**

Symmetry of Odd and Even functions

From our previous discussion of symmetry, $f(-x) = f(x)$ implies that if we replace the x-coordinate by $-x$, the y-coordinate remains the same. This is the equivalent of saying that if (x, y) is on the graph of f, $(-x, y)$ is also on the graph. Therefore, *the graph of an even function is symmetric to the y-axis.*

Similarly, $f(-x) = -f(x)$ implies that if we replace the x-coordinate by $-x$, the y-coordinate becomes $-y$. This is the equivalent of saying that if (x, y) is on the graph of f, $(-x, -y)$ is also on the graph. Therefore, *the graph of an odd function is symmetric to the origin.*

Note that the graph of f cannot be symmetric to the x-axis, because then it would not be a function (It would fail the vertical line test for a function; for values of x, there would be two values of y).

Summarizing

1. **If f is even, that is, if $f(-x) = f(x)$, then the graph of f is symmetric to the y-axis.**

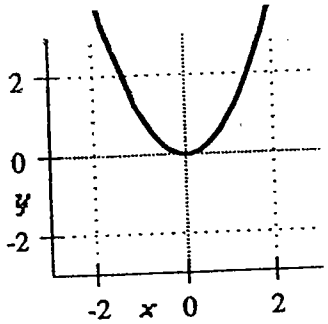
2. **If f is odd, that is, if $f(-x) = -f(x)$, then the graph of f is symmetric to the origin.**

3. **It is possible for a function to be neither even nor odd.**

EXAMPLES OF EVEN FUNCTIONS

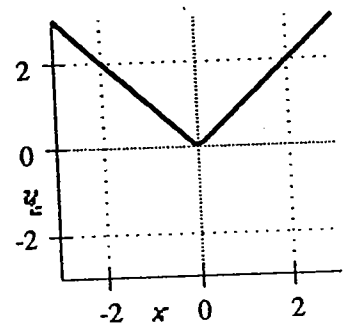
1. $f(x) = x^2$

   $f(-x) = (-x)^2 = x^2 = f(x)$



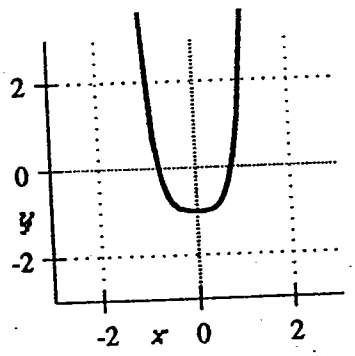
2. $f(x) = |x|$

   $f(-x) = |-x| = |x| = f(x)$



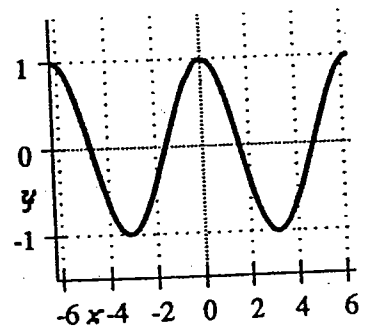
3. $f(x) = 3x^4 - 1$

   $f(-x) = 3(-x)^4 - 1 = 3x^4 - 1 = f(x)$



4. $f(x) = \cos(x)$

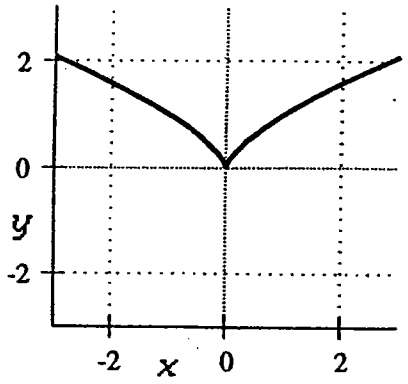
   $f(-x) = \cos(-x) = \cos(x) = f(x)$



5.

$f(x) = x^{2/3} = \sqrt[3]{x^2}$

$f(-x) = \sqrt[3]{(-x)^2} = \sqrt[3]{x^2} = f(x)$

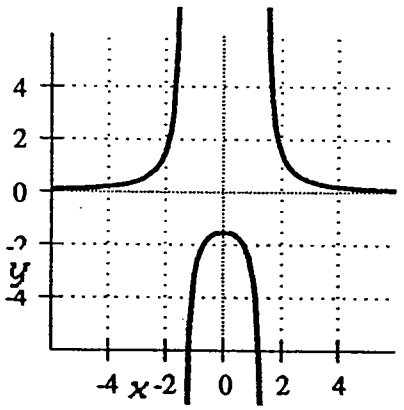


6.

$$f(x) = \frac{3x}{x^3 - 2x}$$

$$f(-x) = \frac{3(-x)}{(-x)^3 - 2(-x)} = \frac{-3x}{-x^3 + 2x}$$

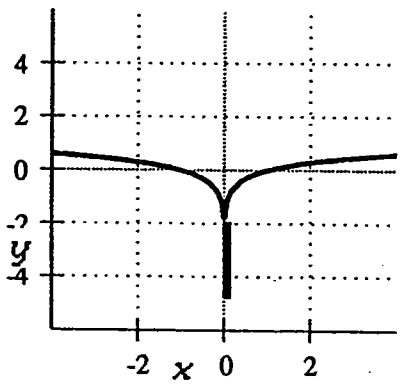
$$= \frac{-3x}{-(x^3 - 2x)} = \frac{3x}{x^3 - 2x} = f(x)$$



7.

$f(x) = \log|x|$

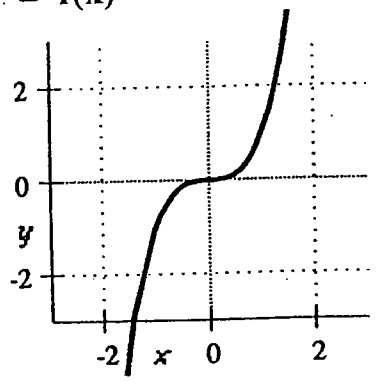
$f(-x) = \log|-x| = \log|x| = f(x)$



EXAMPLES OF ODD FUNCTIONS $f(-x) = -f(x)$

1. $f(x) = x^3$

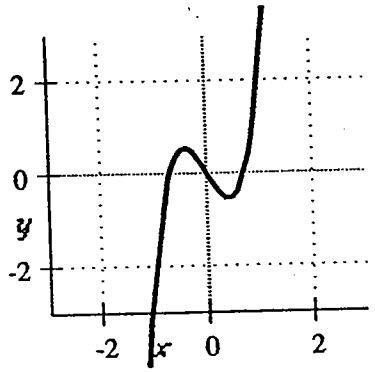
$f(-x) = (-x)^3 = -x^3 = -f(x)$



2. $f(x) = 4x^3 - 2x$

$f(-x) = 4(-x)^3 - 2(-x) = -4x^3 + 2x$

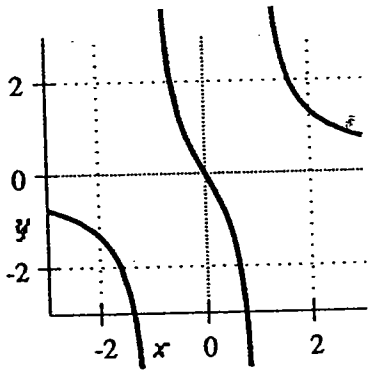
$= -(4x^3 - 2x) = -f(x)$



3. $f(x) = \dfrac{2x}{x^2 - 1}$

$f(-x) = \dfrac{2(-x)}{(-x)^2 - 1} = \dfrac{-2x}{x^2 - 1}$

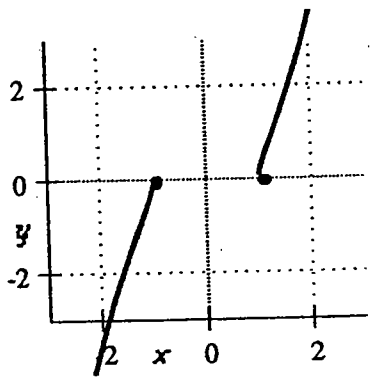
$= -\dfrac{2x}{x^2 - 1} = -f(x)$



4. $f(x) = x\sqrt{x^4 - 1}$

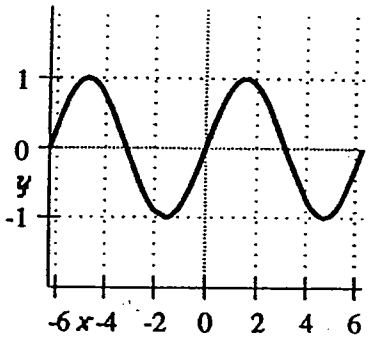
$f(-x) = (-x)\sqrt{(-x)^4 - 1}$

$= -x\sqrt{x^4 - 1} = -f(x)$



5. $f(x) = \sin(x)$

$f(-x) = \sin(-x) = -\sin(x) = -f(x)$

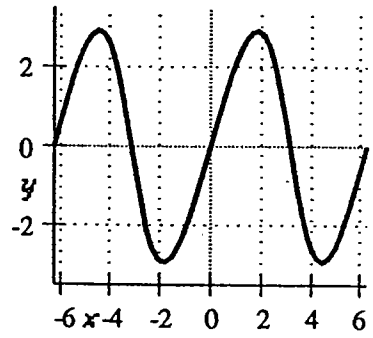


6.

$$f(x) = \frac{4\sin(x)}{\sqrt{2+\cos(x)}}$$

$$f(-x) = \frac{4\sin(-x)}{\sqrt{2+\cos(-x)}} = \frac{-4\sin(x)}{\sqrt{2+\cos(x)}}$$

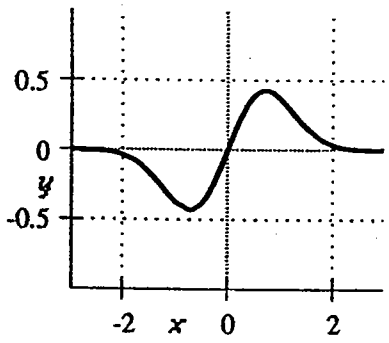
$$= -\frac{4\sin(x)}{\sqrt{2+\cos(x)}} = -f(x)$$



7.

$$f(x) = x\,e^{-x^2}$$

$$f(-x) = (-x)\,e^{-(-x)^2} = -x\,e^{-x^2} = -f(x)$$



POLYNOMIAL FUNCTIONS

If f(x) is a polynomial function, i.e.,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where n is an integer ≥ 0, then

1. **f is an even function if all the exponents of the x-terms are even.**
2. **f is an odd function if all the exponents of the x-terms are odd.**
3. **f is neither even nor odd if the exponents of the x-terms are a mixture of even and odd integers.**

EXAMPLE: Determine whether each of the following polynomial functions are even, odd, or neither.

a. $f(x) = 3x^4 - 2x^2 + 7$

Note that the constant term 7 is equivalent to $7x^0$.

The exponents of x are all even. Therefore f is even.

$f(-x) = f(x)$

b. $f(x) = 3x^5 - 4x^3 - x$

The exponents of x are all odd. Therefore f is an odd function.

$f(-x) = -f(x)$

c. $f(x) = x^3 + 3x - 5$

The constant term -5 is equivalent to $-5x^0$.

The exponents of x are a mixture of odd and even integers.

Therefore f is neither even nor odd

$f(-x) = (-x)^3 + 3(-x) - 5 = -x^3 - 3x - 5 \neq f(x)$ or $-f(x)$

PRODUCT AND QUOTIENTS OF EVEN AND ODD FUNCTIONS

The rules for the product and quotients of even and odd functions are very similar to the rules for multiplication and division of signed numbers, if we think of an odd function as a negative and an even function as a positive.

1. The product or quotient of two even functions is even.
2. The product or quotient of two odd function is even
3. The product or quotient of an even and odd function is odd.

We prove the first part of Rule 2.

Let f(x) and g(x) be odd functions and let $w(x) = f(x)g(x)$.
Prove that w(x) is an even function.

f(x) is odd implies $f(-x) = -f(x)$. g(x) is odd implies $g(-x) = -g(x)$

$w(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = w(x)$. Therefore w is even.

EXAMPLES: Determine whether each of the following functions is even, odd, or neither.

a. $f(x) = x\sqrt{x^2 + 1}$

b. $f(x) = \dfrac{2x}{x^3 + 4x}$

c. $f(x) = x^3 \sin x$

d. $f(x) = \dfrac{\cos x}{x + 1}$

a. odd times even: f is odd.

b. odd divided by odd: f is even

c. odd times odd: f is even

d. even (cos x) divided by a function neither even nor odd.
f is neither even nor odd.

SYMMETRY OF THE ELEMENTARY TRIGONOMETRIC FUNCTIONS

Of the six elementary trigonometric functions, 2 are even, the cosine and its reciprocal, the secant. The other 4 are odd. This implies that

$\cos(-x) = \cos(x)$	symmetry to y-axis
$\sec(-x) = \sec(x)$	"
$\sin(-x) = -\sin(x)$	symmetry to origin
$\csc(-x) = -\csc(x)$	"
$\tan(-x) = -\tan(x)$	"
$\cot(-x) = -\cot(x)$	"

Remembering which functions are even and which are odd is a handy way of remembering the trigonometric identities above.

THE FUNCTION f(|x|)

The function f(|x|), formed by everywhere replacing x in f(x) by |x|, is *always* an even function, regardless of whether f is even, odd, or neither. This is because f(|-x|) = f(|x|) for all x.

EXAMPLES:

f(x) = sin(x) is odd, but g(x) = f(|x|) = sin |x| is even.

$f(x) = x^4 + 3x^2 + 5$ is even, and so is $g(x) = |x|^4 + 3|x|^2 + 5$.

f(x) = x + 1 is neither even nor odd, but g(x) = f(|x|) = |x| + 1 is even.

The concept of even and odd functions will be important later on in calculus. Among other things, calculations involving the definite integral can be simplified if the function is even or odd.

TRANSFORMATIONS

Knowing the graphs of the elementary functions, we can quickly sketch the graphs of many other functions by recognizing them as transformations of the plane, either translations or reflections.

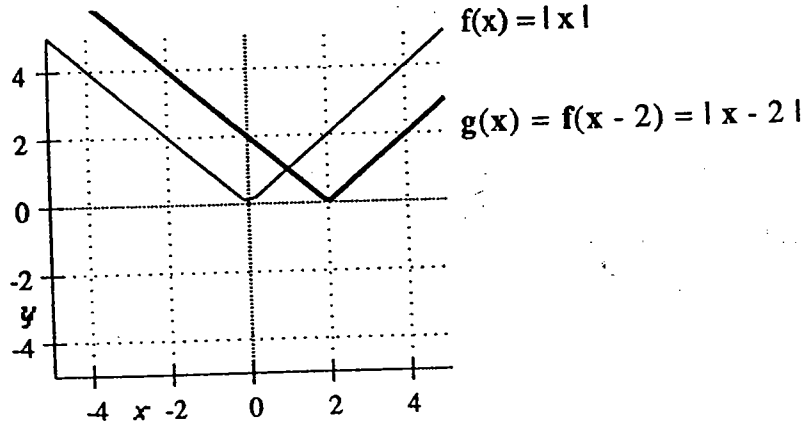
Horizontal Translations
The Graph of f(x - h)

The graph of f(x - h), h a constant, is a horizontal translation of the graph of f(x) by h units, to the right if $h > 0$ (h positive), to the left if $h < 0$ (h negative).

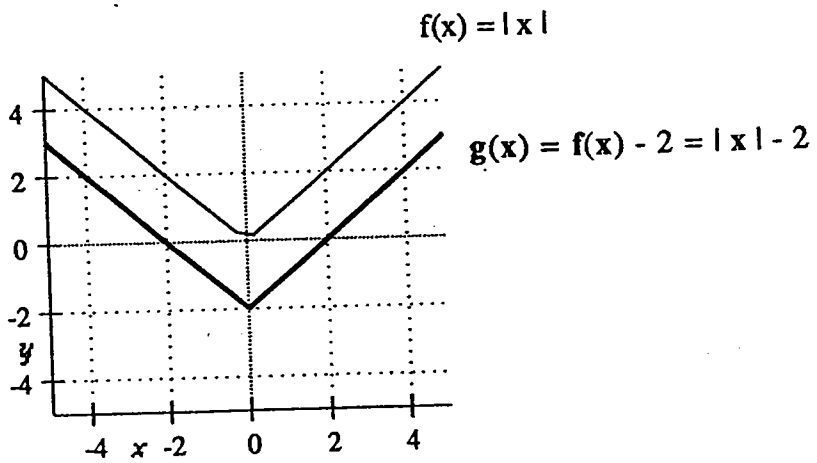
Vertical Translations
The graph of f(x) + k

The graph of f(x) + k, k a constant, is a vertical translation of the graph of f(x) by k units, up if $k > 0$, down if $k < 0$.

$g(x) = f(x - 2)$ Horizontal translation 2 units to the right.

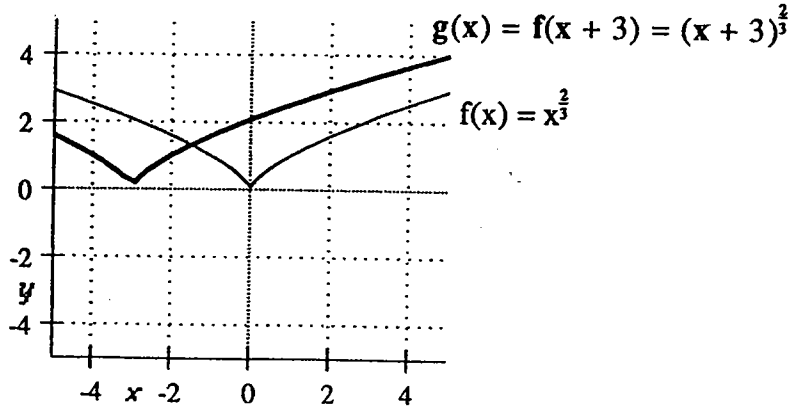


$g(x) = f(x) - 2.$ Vertical translation 2 units down.

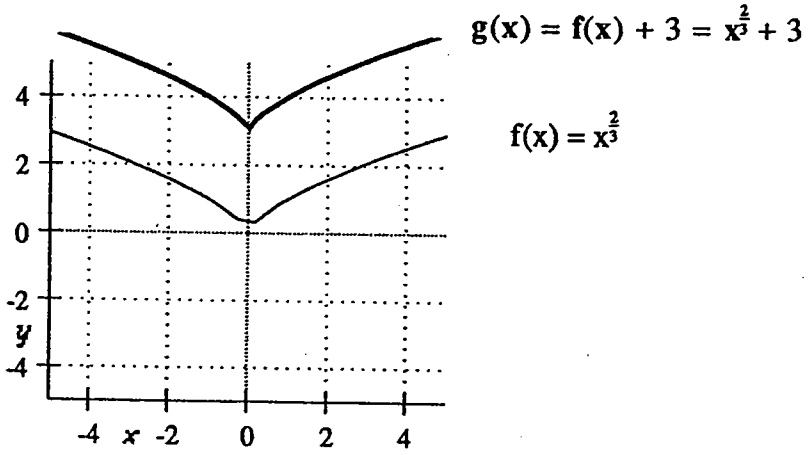


Note the difference between a horizontal and a vertical translation. For a horizontal translation, we replace x by x - h, whereas for a vertical translation we simply add a constant k to the original function.

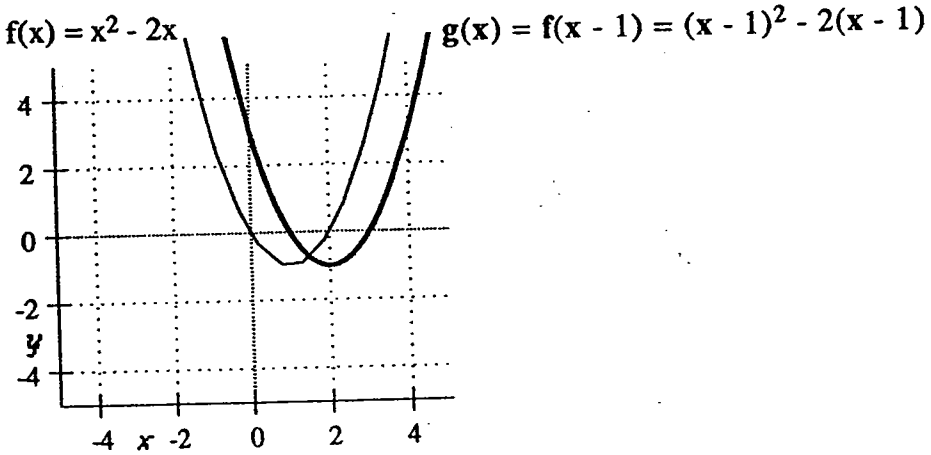
$g(x) = f(x + 3) = f(x - -3)$. *h is negative*. Horizontal translation 3 units to the left.



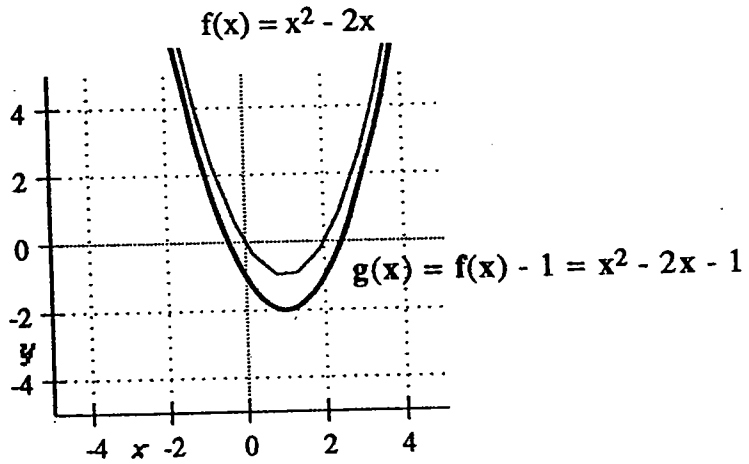
$g(x) = f(x) + 3$. Vertical translation 3 units up.



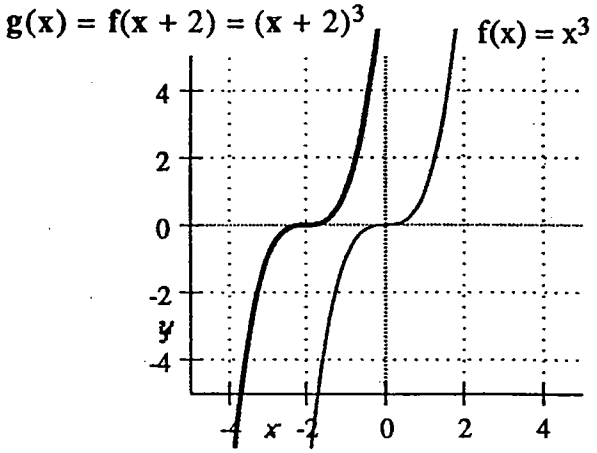
$g(x) = f(x - 1)$. Horizontal translation 1 unit to the right.



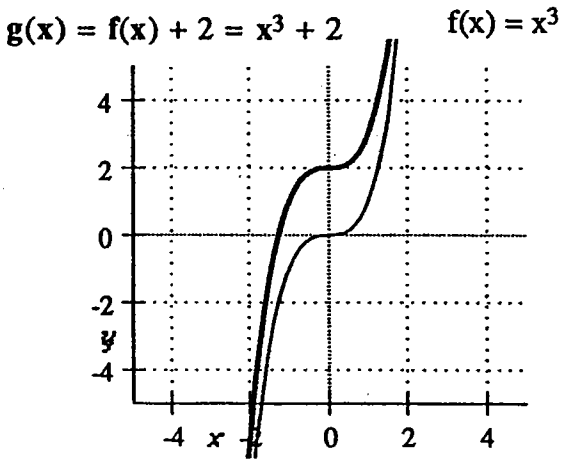
$g(x) = f(x) - 1$. Vertical translation 1 unit down.



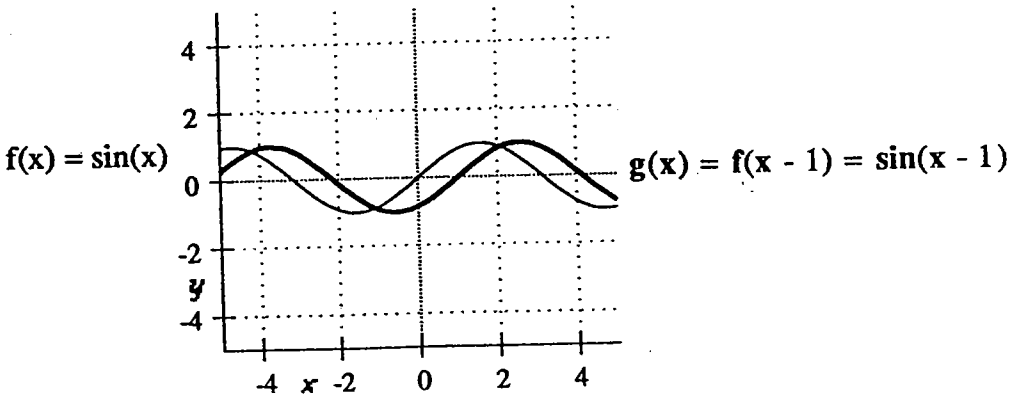
$g(x) = f(x + 2) = f(x - -2)$ Horizontal translation 2 units to the left.



$g(x) = f(x) + 2$. Vertical translation 2 units up.



$g(x) = f(x - 1)$ Horizontal translation 1 unit to the right.



$g(x) = f(x) - 1$ Vertical translation 1 unit down.